On Exponential Changes of Measure for the Feller Diffusion and Superprocesses

by

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Abstract

Let η be a superprocess under the measure P. We show the existence of probability measures which are absolutely continuous with respect to P, and whose Radon-Nikodym derivatives are suitably normalized exponential functions of the self intersection local time of η. These measures correspond to measure valued processes exhibiting a certain amount of self interaction. A finite time divergence of the total mass $\langle 1, \eta_t \rangle$ is shown to occur in a related model in which the change of measure involves the occupation measure of the superprocess. As an independently interesting side issue we also obtain a number of results related to a self-interacting version of the Feller diffusion.

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1 Introduction and main results

1.1 On interactions

Our aim in this paper is to study the effect of certain exponential changes of measure for inducing an internal interaction on the behaviour of superprocesses.

The basic object of study will be a super Brownian motion (SBM), $\{\eta_t\}_{0\le t\le T}$, with $T \ge 0$, and finite. SBM arises naturally via an infinite system of branching particles that alternately perform Brownian motion and undergo critical branching, for a detailed description of which we refer the reader to Dawson (1993) and the references therein. While we shall be motivated by, and use the language of, this particle system (and the relation between additive functionals on the particles and the superprocess as in Adler 1992) we shall, throughout the paper, work exclusively with the limiting superprocess.

The (self) interaction that we wish to apply to the superprocess is one which will cause the "particles" to be attracted either to one another, or to a particular region of space. Broadly speaking, there are two ways in which to do this.

The first, which we shall not develop here, but is interesting in order to understand some of our results, follows from the fact that SBM can be expressed via the solution of a stochastic partial differential equation, describing the time and space flow of particles. This is particularly well described via the so-called "historical process" of Dawson and Perkins (1991) and Perkins (1992).

In this setting it is possible to modulate the diffusions of individual particles, by having them attract, or repel, one another, much in the way that the generator of a finite system of independent diffusions is adapted to yield interaction. This approach was developed in Perkins (1992), and extended to more singular interactions in Adler and Tribe (1996a,b). In all of these cases, it is well known that the new processes are singular (in terms of the probability measures they induce on path space) with respect to the original SBM.

The second approach, which we adopt in the current paper, involves finding more delicate changes that leave the new process absolutely continuous with respect to the old. Attempts to do this via extremely delicate changes of the diffusion seem to be doomed to failure. (cf. Adler and Ivanitskaya 1996 for an example.) Thus we adopt an approach developed initially for defining a self-avoiding Brownian motion (Varadhan 1969, Westwater 1980, 1982). The idea here is to look at the process over some fixed time interval $[0, T]$, and make one global change of measure that favours interacting paths. Note that this is quite different from the above approach, since under this new measure the new process will no longer be Markovian. It is also

different from previous changes of measure dating back to Dawson (1978) in which only the branching mechanism of the superprocess was affected, and not the particle motions.

To describe our approach more precisely we need some notation.

1.2 The superprocess and the Feller diffusion

With $T > 0$, let η be a canonical process on the space $D([0,T], M_f(\Re^d))$, where $M_f(\Re^d)$ denotes the space of finite measures on $\Re^d$. For any $m \in M_f(\Re^d)$, and α real, let $\tilde{P}_{m,T,\alpha}$ be a probability measure on $D([0,T], M_f(\Re^d))$ such that under $\tilde{P}_{m,T,\alpha}$ the process η is Markov and its distribution is characterised by

$$\tilde{E}_{m,T,\alpha} e^{-<f,\eta_t>} = e^{-\langle u_t^{f,\alpha}, m\rangle}, \tag{1.1}$$

where $\tilde{E}_{m,T,\alpha}$ denotes the expectation under $\tilde{P}_{m,T,\alpha}$, $f \in B_+$, the space of non-negative, bounded, measurable functions on $\Re^d$, and u is the unique strong solution on $[0,T]$ of the partial differential equation

$$\frac{\partial u_t}{\partial t} = (\Delta + \alpha) u_t - u_t^2, \tag{1.2}$$

$$u_0 = f. \tag{1.3}$$

Here Δ is the d-dimensional Laplacian. We shall write $\tilde{P}$ and $\tilde{E}$ for $\tilde{P}_{m,T,0}$ and $\tilde{E}_{m,T,0}$ respectively.

When $\alpha = 0$ in the above, the resulting measure valued process is super Brownian motion (SBM). If $\alpha > 0$ ($\alpha < 0$) the corresponding process is SBM with additional, constant rate, mass creation (annihiliation). The reason for adding this extra mass term will become clear later.

We denote by X_t the total mass of the process η_t, that is

$$X_t = \langle 1, \eta_t \rangle = \eta(\Re^d).$$

As is well known, X_t itself is a nice Markov process on $D([0,T], \Re)$, generally known as the "Feller diffusion" and satisfying the stochastic differential equation

$$dX_t = \alpha\, dt + \sqrt{X_t}\, dW_t, \tag{1.4}$$

with $X_0 = \langle 1, \eta_0 \rangle = \langle 1, m \rangle > 0$, and W a standard Brownian motion.

If $\alpha = 0$, then X dies (i.e. reaches zero and stays there) in finite time. If $\alpha > 0$ then X is positive for all $t \geq 0$, but a.s. finite for each t. If $\alpha < 0$, then, since X can become negative, we shall stop it at $\tau_0 := \inf\{t : X_t = 0\}$.

We denote by $P_{x,T,\alpha}$ the distribution of X induced by $\tilde{P}_{m,T,\alpha}$, where $\langle 1, m \rangle = x$ and denote by $E_{x,T,\alpha}$ the corresponding expectation operator.

1.3 Change of measure via SILT

In this subsection, set $\alpha = 0$ in (1.1), so that we are dealing with a standard SBM. We make the following assumption for the whole of this subsection:

Assumption: In addition to being finite, the initial measure m has a bounded density with respect to Lebesgue measure.

For any $\epsilon > 0$, an approximate self-intersection local time (SILT) for the superprocess η can be formally defined as

$$J_T^\epsilon = \int_0^T dt \int_0^T ds\, 1_{\{|t-s|>\epsilon\}} \langle \delta(x-y), \eta_s(dx)\eta_t(dy) \rangle,$$

where δ is the Dirac delta function. J^ϵ is well defined for $d \leq 7$ (Dynkin, 1988). There is no problem sending $\epsilon \longrightarrow 0$ in dimensions $d \leq 3$, and renormalised versions for this limit exist for $d = 4, 5$. (Adler 1992, Rosen 1992).

We can now turn to our first change of measure. Motivated by the true self-avoiding Brownian motion (or "polymer process") we would therefore like to consider, for $d \leq 7$, the probability measure on $D([0,T], M_f(\Re^d))$, defined by

$$\tilde{Q}_{m,T}^{\epsilon,\lambda}(dw) = \frac{e^{\lambda J_T^\epsilon}}{\tilde{E}(e^{\lambda J_T^\epsilon})} \tilde{P}(dw), \quad \lambda > 0. \tag{1.5}$$

Unfortunately, however, the following is true:

Lemma 1.1 *For all $\epsilon, T > 0$, we have that $\tilde{E}_m(e^{\lambda J_T^\epsilon}) \equiv \infty$.*

The proof of the Lemma, which depends on computing the moments of J_T^ϵ, is defered, along with all other proofs, to the following section.

As a consequence of Lemma 1.1, there is no way to work directly with the definition in (1.5). One possible way around this, that was successfully adopted in Adler and Iyer (1997) to construct a self-attracting Brownian motion, is to consider the sequence of measures

$$\tilde{Q}_{m,T}^{n,\epsilon,\lambda}(dw) = \frac{\sum_{j=0}^n \frac{\lambda^j}{j!}(J_T^\epsilon)^j}{\tilde{E}(\sum_{j=0}^n \frac{\lambda^j}{j!}(J_T^\epsilon)^j)} \tilde{P}(dw), \qquad n \geq 1. \tag{1.6}$$

and approach the measure (1.5) via weak convergence as $n \longrightarrow \infty$, and, perhaps, also as $\epsilon \longrightarrow 0$.

Dynkin (1988) has shown that $\tilde{E}(J_T^\epsilon)^2$ is finite for $d \leq 7$. Conditions for the existence of the second moment of the approximate SILT of order n are also given in the above paper. Similar moment calculations can also be found in Rosen (1992) and Feldman and Iyer (1996). A proof of the fact that $\tilde{E}(J_T^\epsilon)^j$ is finite for $d \leq 7$ and all $j \geq 1$ is given in the Appendix.

The main result of this paper, which indicates that a program as described above should work, is

Theorem 1.2 *The collection of probability measures $\{\tilde{Q}_{m,T}^{n,\epsilon,\lambda}\}_{\epsilon>0,n\geq 1}$ is tight in dimensions $d \leq 7$.*

Thus, given $\{n_k, \epsilon_k\}_{k\geq 1}$, the sequence of measures $\{\tilde{Q}_{m,T}^{n_k,\epsilon_k,\lambda}\}$ contains at least one subsequence that converges weakly in $D([0,T], M_f(\Re^d))$, that may be used to define a limiting polymer measure. We shall have nothing to say about the uniqueness (or otherwise) of this limit. However, we shall investigate a few of its properties.

Comment for Robert We may have something to say about uniqueness.

Of course, while Theorem 1.2 gives us a candidate for a new polymer process, it does not tell us that the new process is any different from the old. (This is a non-trivial issue, in view, for example, of the aforementioned problems with interesting looking interactions at the particle level that disappear in the infinite density limit.) The following result indicates that the polymer measure is quite different from the initial superprocess measure, in at least a number of ways.

Theorem 1.3 *a. For all $\lambda > 0$,*

$$\lim_{\epsilon\downarrow 0} \tilde{E}_{m,T}^{1,\epsilon,\lambda}(X_t) = \begin{cases} |m|(1 + 2\lambda\frac{h_1(d,T,t)}{1+\lambda h_2(d,T,m,|m|)}) & 1 \leq d \leq 3, \\ |m| + \frac{T^2-(T-t)^2}{T} & 4 \leq d \leq 7 \end{cases} \tag{1.7}$$

where h_1 and h_2 are positive, bounded functions.

b. For all $\lambda > 0$ and $4 \leq d \leq 7$

$$\lim_{n\to\infty}\lim_{\epsilon\downarrow 0} \tilde{E}_{m,T}^{n,\epsilon,\lambda}(X_t) = \tilde{E}_m(X_t)\left(1 + \frac{T}{|m|}\right) \quad t > 0. \tag{1.8}$$

Note firstly that for the basic SBM we have $\tilde{E}(X_t) = |m|$, for all $t > 0$, and all d. Thus, (1.7), (1.8) tell us that not only is the polymer measure different from that of SBM, but it also involves "creation of mass". Looking back at the formulation (1.6)

of $\tilde{Q}^{n,\epsilon,\lambda}_{m,T}$, and, more simply if not so precisely (1.5), it is clear that, under $\tilde{Q}$, SILT is enhanced. In the development of a regular polymer measure for standard Brownian motion this means that the particle path must be changed. Here, however, there are two ways to increase SILT: One is by moving the paths, the other is by affecting the branching so that the total mass (and so SILT) is increased. Theorem 1.3 indicates that the latter is certainly happenning here. How much interference with the particle motion in being generated by the change of measure, and how the two effects can be uncoupled, is currently unclear.

Comment for Robert Another point to be made is that when we first send ϵ to zero and then send n to infinity, the expected value of the total mass of the limiting process has a jump at $t = 0$, and then it stays constant as opposed to the case of fixed n, when there is a continuous increase in the expected total mass. This suggests that things may be different in the limit. This is similar/a contrast to the fact that even if we start with an absolute continuous mass the SBM immediately becomes singular.

A second rather interesting consequence of (1.7) is that, at least for dimensions $4 \leq d \leq 7$, we do not recover the basic SBM by sending the interaction strength parameter, λ, to 0.

As an interesting aside, note that even the case of fixed n and $\epsilon = 0$ gives a measure different from that of SBM. This result is at variance with the result for the case of the attractive polymer model for the Brownian motion (admittedly a different – but similar looking – problem) where keeping n fixed and sending ϵ to zero lead us back to the Brownian motion (Adler and Iyer 1997).

Theorem 1.4 *The underlying Feller (total mass) process is uniquely determined under the limiting probability measure* $\lim_{n\to\infty} \lim_{\epsilon\downarrow 0} \tilde{Q}^{n,\epsilon,\lambda}_{m,T}$, *for all* $T > 0$ *and* $\lambda > 0$ *in dimensions* $4 \leq d \leq 7$.

Despite the fact that J^{ϵ}_{T} is not exponentially integrable, we do not lose the exponential integrability of the basic superprocess in the following particular case.

Theorem 1.5 *Let* $4 \leq d \leq 7$ *and* $f \in B_{+}$. *Then the limits of the sequence* $\tilde{E}^{n,\epsilon,\lambda}_{m,T}(e^{\langle f,\eta_t\rangle})$ *are finite when we first let* $\epsilon \to 0$ *and then take the limit* $n \to \infty$, *for* $t < (C\|f\|_{\infty})^{-1}$, *where* $C = 1 + 2(1 \vee |m|^{-1})(1 \vee T)$.

1.4 Change of measure via occupation measure

Given the fact that when changing the probability via SILT we (inadvertently?) introduced a mass creation term, it seems interesting to look at a simpler model as well.

Thus, in this section we shall introduce interactions in such a way that the process favours more mass in certain locations of the space.

For $\phi \in B_+$, define the occupation measure (or "man hours" cf. Iscoe 1986a,b) process

$$\mu_t(\phi) \equiv \langle \phi, \mu_t \rangle := \int_0^t \langle \phi, \eta_s \rangle \, ds \tag{1.9}$$

and consider the new measure defined by

$$\tilde{Q}^{\lambda,\phi}_{x,T,\alpha}(dw) := \frac{e^{\lambda \mu_T(\phi)}}{\tilde{E}(e^{\lambda \mu_T(\phi)})} \tilde{P}_{x,T,\alpha}(dw), \qquad \lambda > 0. \tag{1.10}$$

If, for example, ϕ has compact support A, then what we have defined is a measure valued process that favours realisations putting most of their mass in A. Again, what is not clear is whether this is done by redirecting the individual particles to drift towards A, or by generating mass creation in A. In SILT model we saw an increase in total mass, but could not say much beyond that.

However, if we take $\phi \equiv 1$ then some simple analysis of the total mass process (i.e. the Feller diffusion) shows that the mass creation is so intense that in general the process will explode in fixed time. This is described below in the following section, which is essentially independent of the remainder of the paper.

A more careful analysis of general ϕ, and, in particular, of the particularly interesting case when $\phi = \delta$, so that the occupation time is replaced by the local time, will be the object of a future study.

1.5 The reinforced Feller diffusion

As before, let $X_t = \langle 1, \eta_t \rangle$ be the total mass process: i.e. the Feller diffusion (1.3) and $\mu_t := \int_0^t X_s ds = \mu_t(1)$. Consider the following change of measure, which, along with the remainder of this subsection, has nothing to do with superprocesses:

$$P^{\lambda}_{x,T,\alpha}(dw) = \frac{e^{\lambda \mu_T}}{E(e^{\lambda \mu_T})} P_{x,T,\alpha}(dw), \qquad \lambda > 0. \tag{1.11}$$

Since this change of measure clearly defines a process that likes to take high values we shall call it a *reinforced Feller diffusion*. (Of course, the term "diffusion" here is somewhat inappropriate, since this process is not Markovian!)

It is actually possible to explicitly compute the Laplace transform of X_t for $0 \le t \le T$, under the new measure, and the result is surprisingly dependent on the values of α and λ. Here are some sample results:

Theorem 1.6 *Let $\lambda, c > 0$, and set $\gamma = \lambda - \alpha^2/4$. Then, for all $0 < t < T$,*

(i)

$$E^{\lambda}_{x,T,0}(e^{-cX_t}) = \frac{\exp\{\sqrt{\lambda}\, x \tan[\sqrt{\lambda}t - \tan^{-1}(\frac{c-\sqrt{\lambda}\tan((T-t)\sqrt{\lambda})}{\sqrt{\lambda}})]\}}{\exp\{\sqrt{\lambda}\, x \tan(\sqrt{\lambda}T)\}} \tag{1.12}$$

(ii) If $\gamma > 0$, then

$$E^{\lambda}_{x,T,\alpha}(e^{-cX_t}) = \frac{\exp\{-\sqrt{\gamma}\, x \tan[\tan^{-1}(\frac{c-\sqrt{\gamma}\tan(\tan^{-1}(\frac{\alpha}{2\sqrt{\gamma}})+\sqrt{\gamma}(T-t))}{\sqrt{\gamma}} - \sqrt{\gamma}t]\}}{\exp\{\sqrt{\gamma}\, x \tan[\tan^{-1}(\frac{\alpha}{2\sqrt{\gamma}}) + \sqrt{\gamma}T]\}} \tag{1.13}$$

(iii) If $\gamma = 0$, then

$$E^{\lambda}_{x,T,\alpha}(e^{-cX_t}) = \frac{\exp\{-\frac{c(1-\frac{\alpha}{2}(T-t))-\frac{\alpha}{2}}{1-\frac{\alpha}{2}(T-t)+t(c(1-\frac{\alpha}{2}(T-t))-\frac{\alpha}{2})}\, x\}}{\exp\{\frac{\frac{\alpha}{2}}{1-\frac{\alpha}{2}T}\, x\}} \tag{1.14}$$

(iv) If $\gamma < 0$, let $\beta = \sqrt{-\gamma}$. Then

$$E^{\lambda}_{x,T,\alpha}(e^{-cX_t}) = \frac{\exp\{-w_t^{c+w^0_{T-t}}\, x\}}{\exp\{-w^0_T\, x\}}, \tag{1.15}$$

where,

$$w^c_t - \frac{\alpha}{2} = \beta\left[\frac{\beta(e^{2\beta t}-1) + (c-\frac{\alpha}{2})(e^{2\beta t}+1)}{\beta(e^{2\beta t}+1) + (c-\frac{\alpha}{2})(e^{2\beta t}-1)}\right]. \tag{1.16}$$

An unexpected corollary of the above is that, under the new measure, if T approaches a particular (α, λ dependent) time, mass creation is so fierce that the corresponding processes diverge to $+\infty$ in finite time. More precisely,

Corollary 1.7 *(Explosion of mass) Let γ be as in the previous theorem. Let $\lambda > 0$.*

(i) $E^{\lambda}_{x,T,0}(e^{-cX_t}) \to 0$ as $T \uparrow \frac{\pi}{2\sqrt{\lambda}}$.

(ii) If $\gamma > 0$, then $E^{\lambda}_{x,T,\alpha}(e^{-cX_t}) \to 0$ as $T \uparrow (\frac{\pi}{2} - \tan^{-1}(\frac{\alpha}{2\sqrt{\gamma}}))\frac{1}{\sqrt{\gamma}}$.

(iii) If $\alpha > 0$ and $\gamma = 0$, then $E^{\lambda}_{x,T,\alpha}(e^{-cX_t}) \to 0$ as $T \uparrow 2/\alpha$.

(iv) If $\gamma < 0$ and $\alpha > 2\beta$, then $E^{\lambda}_{x,T,\alpha}(e^{-cX_t}) \to 0$ as $T \uparrow \frac{1}{2\beta}\ln\left(\frac{\frac{\alpha}{2}+\beta}{\frac{\alpha}{2}-\beta}\right)$.

A number of additional properties of the reinforced Feller diffusion are given in Section 3 below. While they seem to be of independent interest, and so are included there, they are not related to the main theme of this paper.

2 Proofs and supporting results

Given the form of our change of measure, it should come as no surprise that most of the proofs of the results in subsection 1.3 will be based on an analysis of the moments of SBM.

Dynkin (1988) derived what is now a well known explicit representation for these moments in terms of the transition densities $p_t(x, y)$ and binary graphs. Since his representation will form the backbone of the proofs, we include it here for the sake of completeness.

2.1 Moment Formulae for the Superprocess

Let $\mathcal{D}_k$ be the set of all directed binary graphs with k exits marked $1, 2, \ldots, k$. Given such a graph G, let L be the set of directed links and V be the set of all vertices. If the link $l \in L$ goes from vertex $v \in V$ to vertex $w \in V$, we write $v = i(l), w = f(l)$. We associate with each element $v \in V$ two variables,

$$(s_v, y_v) \in \Re_+ \times \Re^d,$$

which we refer to as the time and space coordinates of v. We denote by V_- the set of all entrances of G; i.e. the set of all vertices $v \in V$ such that no arrow ends at v and there is only one arrow that begins at v. For any $v \in V_-$ we set $s_v = 0$ and $y_v = x_v$. If v is the exit labeled by j, $1 \le j \le k$ – that is, v is such that there is exactly one arrow which ends at v and no arrows begin at v – we set

$$(s_v, y_v) \equiv (t_j, z_j).$$

Finally, let V_o denote the set of internal vertices; i.e. those vertices that are neither entrances nor exits.

Let $p_t(x, y) = P_t(x - y) = (2\pi t)^{-d/2} \exp(-\|x - y\|^2/2)$, $t > 0$, be the standard Brownian transition density, and set $p_t \equiv 0$ if $t < 0$.

Theorem 2.1 *Let η be a super Brownian motion. Then, for all $k \ge 1$, $t_1, \ldots, t_k \ge 0$ and $f_1, \ldots, f_k \in B_+$,*

$$E\left(\prod_{i=1}^{k} \langle f_i, \eta_{t_i} \rangle\right) = \sum_{D \in \mathcal{D}_k} C_D, \tag{2.1}$$

where

$$C_D = \int \prod_{v \in V_-} m(dx_v) \prod_{l \in L} p_{s_{f(l)} - s_{i(l)}}(y_{f(l)} - y_{i(l)}) \prod_{v \in V_o} ds_v dy_v \prod_{i=1}^{k} f_i(z_i) dz_i. \tag{2.2}$$

For detailed examples of how these formulae are applied see, for example, the original paper of Dynkin (1988) and Rosen (1992). Adler (1992) explains the relationship of Theorem 2.1 to the particle picture.

2.2 Preliminary (but crucial) estimates

In most of the calculations that follow, we shall be interested in deriving moment estimates in which the crucial issue will be identifying (negative) powers of the parameter ϵ. It can be seen from the proof of Theorem 1.3 that the exact computation of the moments are both extremely involved, and, fortunately, unnecessary for our purposes. Since most of the techniques involved in estimating these moments have already been explored in great detail in Dynkin (1988) and Rosen (1992), the proofs tend to be a bit sketchy at times, although we do hope that we have covered all the important points.

We start with some general moment bounds, in which $f(\epsilon) \approx g(\epsilon)$ means that there exists a constant c such that $\lim_{\epsilon\to 0} f(\epsilon)/g(\epsilon) = c$.

Lemma 2.2 *For* $d \leq 7$,

$$\tilde{E}_m(J_T^\epsilon)^j \approx \sum_{D\in\mathcal{D}_{2j}} C(m,d,D)(\ln T)^{\gamma(D)} T^{-\frac{c(D)d}{2}+a(D,d)} b(d,D,\epsilon), \tag{2.3}$$

where $0 \leq k(D) \leq j$, $0 \leq c(D) \leq j - k(D)$, *and*

$$a(D,d) = \begin{cases} |D| & 1 \leq d \leq 3 \\ |D| - 2k(D) & 4 \leq d \leq 7 \end{cases} \tag{2.4}$$

$$b(d,D,\epsilon) = \begin{cases} 1 & 1 \leq d \leq 3 \\ (\ln(\epsilon))^{k(D)} & d = 4 \\ \epsilon^{-(d-4)k(D)/2} & 5 \leq d \leq 7 \end{cases} \tag{2.5}$$

$\gamma(D)$ is non-negative if d is even and zero otherwise. While we will indicate were this extra logarithmic term in (2.3) arises when d is even in the proof, we shall avoid the extra algebra involved by treating, in detail, only the cases when d is odd.

Proof of Lemma 2.2. Recall that

$$(J^\epsilon)^j = \prod_{i=1}^{j} \int \langle \delta(x-y), \eta_{t_{2i-1}}(dx)\eta_{t_{2i}}(dy)\rangle 1_{\{|t_{2i-1}-t_{2i}|>\epsilon\}} dt_{2i-1}dt_{2i} \tag{2.6}$$

$$\tilde{E}_m(J^\epsilon)^j = \sum_{D \in \mathcal{D}_{2j}} C'_D, \tag{2.7}$$

where,

$$C'_D = \prod_{i=1}^{j} \int_0^T \int_0^T 1_{|t_{2i}-t_{2i-1}|>\epsilon} dt_{2i} dt_{2i-1} \int \prod_{v \in V_-} m(dx_v) \prod_{l \in L} p_{s_{f(l)}-s_{i(l)}}(y_{f(l)} - y_{i(l)}) \prod_{v \in V_o} ds_v dy_v \prod_{i=1}^{j} dz_i, \tag{2.8}$$

where on account of the delta function we have $y_{f(l)} = z_i$ if $s_{f(l)} = t_{2i}, t_{2i-1}$.

Note first that for the graphs that contribute to $E(J^\epsilon)^j$ the exit nodes, in the notation of (2.2) are all labelled (t_k, z_i), $k = 1, \ldots, 2j$, $i = 1, \ldots, j$.

We say that two exit nodes (t_k, z_i) and (t_ℓ, z_j) are paired if there is an internal node labelled (s, y) such that there is an edge connecting (s, y) to (t_k, z_i) and (t_ℓ, z_j). Here is a preliminary calculation for paired nodes:

Consider the term corresponding to any graph D. Let the graph D have $1 \leq r \leq 2j$ entrance nodes; i.e. r sub-trees. Let $|D|$ denote the number of edges in D. Suppose for some $k = 1, \ldots, j$ the nodes (t_{2k-1}, z_k) and (t_{2k}, z_k) are paired. (Note that the space variables associated with the exit nodes with time variables t_{2k-1} and t_{2k} are identical as a result of the delta function appearing in the definition of J^ϵ.) We shall call nodes paired this way as *twin pairs*. For twin pairs, integrating with respect to z_k (and applying the Markovian nature of p_t) leaves the external integral involving t_{2k-1} and t_{2k} in the form

$$\frac{1}{(2\pi)^{-d/2}} \int_s^T \int_s^T dt_{2k-1} dt_{2k} 1_{\{|t_{2k}-t_{2k-1}|>\epsilon\}} (t_{2k-1} + t_{2k} - 2s)^{-d/2} = c_1(d, T-s, \epsilon) \tag{2.9}$$

If $d = 4$, then $c_1(d, T-s, \epsilon) \approx \ln(\epsilon)/2$. If $5 \leq d \leq 7$, then $c_1(d, T-s, \epsilon) \approx (4/(d-4)(d-2))\epsilon^{-(d-4)/2}$. Thus, if $4 \leq d \leq 7$, (2.9) diverges as $\epsilon \to 0$ and the important terms are those that diverge in ϵ. If $1 \leq d \leq 3$, then (2.9) is bounded in ϵ and we can take ϵ to be zero, to get $c_1(d, T-s, \epsilon) \approx (2^{(d-6)/2}/(d-4))(T-s)^{-(d-4)/2}$.

With this preparation behind us, we start the main computation for evaluating a typical term in (2.7):

Integrate out all the j–space variables corresponding to the $2j$ exit nodes. For all exit nodes that are twin pairs, also integrate with respect to the time variables as in (2.9). Suppose there are $0 \leq k(D) \leq j$ twin pairs.

Integrate out all the internal space variables as shown in the appendix. This is standard fare. Ultimately we arrive back to the outermost integral with respect to $m(dx)$, and here we have to work a little harder, since while in the appendix (as is the usual case for these computations) we are interested in general upper bounds here we shall require more precision.

Recall the assumption that $m(dx) = f(x)dx$, for some $f \in \mathcal{B}_+(\Re^d)$. A generic outermost term will now look like

$$\int m(dx)p_s(x,y)g_D(s,y)dsdy = \int T_sf(y)g_D(s,y)dsdy, \tag{2.10}$$

where T_t is the heat semigroup and $g_D(s,y)$ is the term obtained by integrating out all the space and time variables, except for the space/time variables corresponding to the first edge from the root in the sub-tree with entrance node labelled $(0,x)$.

¿From the appendix we know that $\int g_D(s,y)dsdy < \infty$. Moreover $g_D(s,y)$ and $T_sf(y)$ are strictly non-negative functions on $[0,T] \times \Re^d$ and $(0,T] \times \Re^d$ respectively. g_D and T_sf are continuous functions on $(0,T] \times \Re^d$. Let K_n be any sequence of compact sets increasing to $\Re^d$. Then, for some sequence (s_n, y_n) (2.10) is equal to

$$\lim_{n\to\infty} \int T_sf(y)g_D(s,y)1_{K_n}(y)dsdy = \lim_{n\to\infty} T_{s_n}f(y_n) \int g_D(s,y)dsdy, \tag{2.11}$$

a fact which follows from the mean value theorem applied to the function $T_sf(y)$ integrated against the measure $g_D(s,y)1_{K_n}(y)dsdy$. Since $0 < T_{s_n}f(y_n) < ||f||_\infty$ we have that $T_{s_n}f(y_n)$ converges to some $c_D \in (0, ||f||_\infty)$ as $n \to \infty$ (since the right side of (2.11) converges to the left side of (2.10)). Consequently,

$$\int m(dx)p_s(x,y)g_D(s,y)dsdy = c_D \int_{[0,T]\times\Re^d} g_D(s,y)dsdy.$$

If we use the above technique judiciously, as in the appendix, we can integrate out all the transition density functions associated with the internal edges while integrating out the internal space variables. Now we can integrate out the remaining time variables corresponding to the remaining exit nodes in a manner similar to (2.9). In evaluating these integrals we can take $\epsilon = 0$. If $\mathrm{s} = (s_{l_1}, \ldots, s_{l_{j-k(D)}})$ are the time variables corresponding to the internal vertices of the graph D that are connected to these exit nodes, we can write this factor as

$$\frac{\prod_{i \le m(D)} (r_i(T,\mathrm{s}))^{e_i}}{c_3(D,d)},$$

where r_i's are linear combinations of T and the components of s, and $\sum e_i = -c(D)d/2 + 2(j-k(D))$, and $0 \le c(D) \le (j-k(D))/2$. Note that each s_{l_i} will occur as an argument in exactly one function r_i. In case d is even, it is here that we will get factors containing logarithms of r_i's. This will make the subsequent calculations very messy.

A note here on the exact value of $c(D)$. This will be used later in the proof of Theorem 1.2. To compute $c(D)$, we integrate as described above all the twin pairs. We can also integrate out the space variables corresponding to the exit nodes for which one of them is on a subgraph with only one edge. Now consider the residual

graph. In this graph, suppose that the exit nodes are labeled $z_1, \ldots, z_p$. We partition $z_1, \ldots, z_p$ into disjoint sets $S_1, \ldots, S_q$ such that the exit nodes in the distinct sets are on different subgraphs. Then, $c(D) = q$. Since of the remaining $j - k(D)$ pairs of nodes there are no twin pairs, the maximum value of $C(D)$ will be as given above. For example, for the graph in figure 5 in appendix $c(D) = 1$. For the graphs $(i), (ii)$ and (v) in figure 4, $c(D) = 0$, whereas for the graphs $(iii), (iv), (vi)$ and (vii) $c(D) = 1$.

This leaves us with $|D| - 2j$ internal time variables that on integration yield a factor $T^{|D|-2j}/c_4(D)$. For example, in the case $r = 2j$, $c_4(D) = 1$. If $r = j$ and all the exit nodes are twin pairs, then there are j internal time variables and $c_4(D) = c^j$. Note that the exponent of T in our estimate will be $-c(D)d/2 + |D| - 2k(D)$, for $4 \leq d \leq 7$, and will equal $-c(D)d/2 + |D|$ for $1 \leq d \leq 3$. In case d is even, there will be additional terms in $\ln(T)$.

We now claim, and will prove below, that

$$j \leq |D| - 2k(D) \leq 4j. \tag{2.12}$$

It is now easy to check that the bounds of the Lemma are obtained by simply putting $\epsilon = 0$ in all the terms that do not diverge as $\epsilon \to 0$. In fact, we can now write the following estimate for $\tilde{E}_m(J^\epsilon)^j$:

$$\tilde{E}_m(J^\epsilon)^j \approx \sum_{D \in \mathcal{D}_{2j}} C(m, d, D) T^{-\frac{c(D)d}{2} + a(D,d)} b(d, D, \epsilon), \tag{2.13}$$

where $b(d, D, \epsilon)$ equals 1 if $1 \leq d \leq 3$, equals $(\ln(\epsilon))^{k(D)}$ if $d = 4$ and is given by $\epsilon^{-(d-4)k(D)/2}$ if $5 \leq d \leq 7$. $k(D)$ is the number of twin pairs in D. $a(D, d)$ is given by $|D|$ if $d = 1, 3$ and equals $|D| - 2k(D)$ if $4 \leq d \leq 7$, and $0 \leq c(D) \leq j$.

This proves Lemma 2.2, except for the inequality (2.12).

Since $|D| - 2k \leq |D| \leq 4j$, the right hand inequality of (2.12) is trivial. For the left hand inequality, observe that we can associate atleast three distinct edges of D with each twin pair, and that there are k twin pairs. Furthermore, with each of the remaining exit nodes numbering $2j - 2k$ we can associate atleast one distinct edge.

Thus, subtracting $2k$ from $|D|$ corresponds to taking into account only one of the three edges associated with the paired nodes, so that

$$|D| - 2k \geq (3k - 2k) + (2j - 2k) = 2j - k > j$$

This proves the claim. ■

2.3 Proofs of the main results

Proof of Lemma 1.1. Since J_T^ϵ is positive, it is enough to bound from below the contribution to $E(J_T^\epsilon)^j$. We do this by restricting our attention to complete binary graphs. The case $j = 2$ of such a complete binary graph is illustrated in (2.3). There

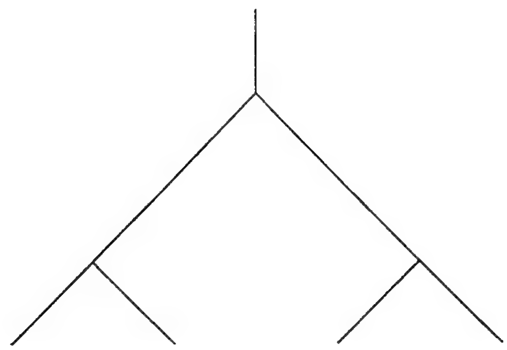

Figure 1: Complete binary graph with 4 exits (j=2)

will be $2j!$ terms associated with a complete binary graph. This, together with the estimate for $E_{m,T}(J_T^\epsilon)^j$ obtained above, gives us that

$$\frac{\lambda^j}{j!}E_{m,T}(J_T^\epsilon)^j \geq \frac{\lambda^j}{j!}C(m,d,D)b(d,D,\epsilon)(T^{-d/2+1})^j(2j)! = O(j^j), \tag{2.14}$$

for large j. This implies that

$$\sum_{j=0}^{n}\frac{\lambda^j}{j!}E_{m,T}(J_T^\epsilon)^j \to \infty, \quad \text{as} \quad n \to \infty. \tag{2.15}$$

This proves Lemma 1.1. ■

Proof of Theorem 1.2. In order to prove tightness of the family of measures $\{\tilde{Q}_{m,T}^{n,\epsilon,\lambda}\}_{n\geq 1,\epsilon>0}$, we start with a bound for $\tilde{E}_{m,T}^{n,\epsilon,\lambda}(\langle f,\eta_t\rangle^k)$. Recall that

$$\tilde{E}_{m,T}^{n,\epsilon,\lambda}(\langle f,\eta_t\rangle^k) = \frac{\sum_{j=1}^{n}\frac{\lambda^j}{j!}\tilde{E}_m(\langle f,\eta_T\rangle^k(J_T^\epsilon)^j)}{\sum_{j=1}^{n}\frac{\lambda^j}{j!}\tilde{E}_m(J_T^\epsilon)^j} \tag{2.16}$$

We will compare $\tilde{E}_m(\langle f,\eta_T\rangle^k(J_T^\epsilon)^j)$ with $\tilde{E}_m(J_T^\epsilon)^j$; i.e. the corresponding terms in the numerator and the denominator. Clearly,

$$\tilde{E}_m(\langle f,\eta_T\rangle^k(J_T^\epsilon)^j) \leq \|f\|^k\tilde{E}_m(\langle 1,\eta_t\rangle^k(J_T^\epsilon)^j) \tag{2.17}$$

Let

$$\tilde{E}_m(J_T^\epsilon)^j = \sum_{D\in\mathcal{D}_{2j}} C_D; \quad \text{and} \quad \tilde{E}_m(\langle 1,\eta_t\rangle^k (J_T^\epsilon)^j) = \sum_{D'\in\mathcal{D}_{2j+k}} C'_{D'}, \tag{2.18}$$

where, as before, $\mathcal{D}_{2j}$ is the set of all binary graphs with $2j$ labelled exits.

Any $D' \in \mathcal{D}_{2j+k}$ can be obtained from a unique $D \in \mathcal{D}_{2j}$ in one of the following two ways:

(1) Add a graph $D'' \in \mathcal{D}_k$ to the graph D as a seperate component. The sum over all graphs $D' \in \mathcal{D}_{2j+k}$ obtained in this way will give a term of the form

$$\tilde{E}_m\langle 1,\eta_t\rangle^k \tilde{E}_m(J_T^\epsilon)^j. \tag{2.19}$$

(2) Take a graph $D'' \in \mathcal{D}_k$. Let D'' have $1 \le r \le k$ components (that is r entrance nodes). We attach $1 \le p \le r$ of the components of D'' to the edges of D. This can be done in

$$\sum_{p=1}^{r}\binom{r}{p}\sum_{l=1}^{p}\left(\sum_{\Lambda(p,l)}\binom{p}{p_1}\binom{p-p_1}{p_2}\cdots\binom{p-\ldots-p_{l-2}}{p_{l-2}}p_1!\cdots p_l!\,a(a-1)\cdots(a-l+1)\right), \tag{2.20}$$

ways, where, $\Lambda(p,l) = \{p_1,\ldots,p_l : p_i > 0, i = 1,\ldots,l, \sum_{i=1}^l p_i = p\}$ and $a = |D|$. Truly speaking, (2.20) is an over estimate, since one has to divide in the innermost sum by the factorials of the number of p_i's that are identical.

Observe that the space variables in the inserted graph D'' can be integrated out, since the function associated with the exit nodes in this graph is the constant function 1. After doing this, the space variables of the p internal nodes at the points where the p components of D'' have been appended to the edges of D can also be integrated out. Next we deal with the time integrals coming from the inserted part. If D'' and D are as illustrated in Figure 2. then, inserting D'' into D on the edge going from the node

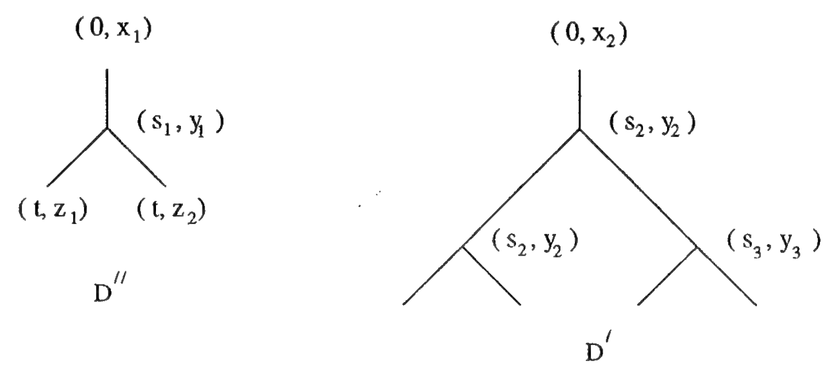

Figure 2: Two graphs pre-insertion

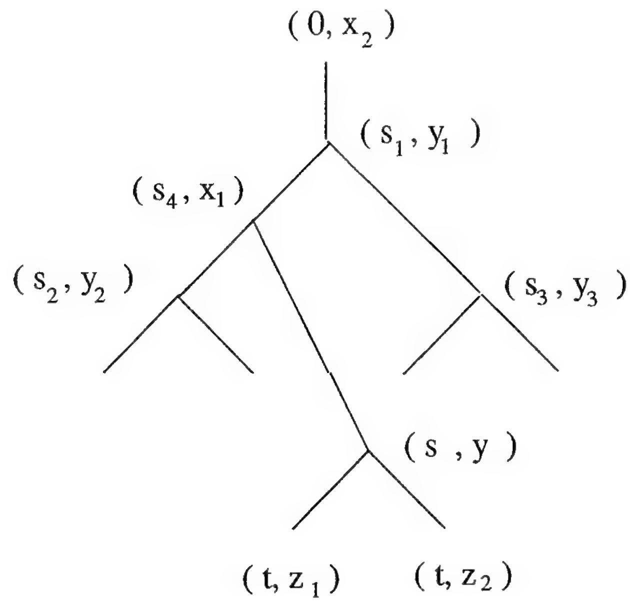

Figure 3: Graphs of Figure 2, post-insertion

labelled (s_1, y_1) to the node (s_2, y_2) gives the graph illustrated in Figure 3. Note that the node $(0, x)$ has been changed to (s_4, x_1). The term corresponding to the graph D'' (considered as a seperate graph) is

$$\int m(dx) \int_0^t ds \int dy\ p_s(x,y) \int p_{t-s}(y,z_1)p_{t-s}(y,z_2)dz_1dz_2 = \mid m \mid \int_0^t ds. \qquad (2.21)$$

Thus the factor corresponding to the inserted part D'' into D as shown in Figure 2.3 will be

$$\int_{s_4}^t ds \leq \int_0^t ds. \qquad (2.22)$$

Consequently, replacing the lower bounds in all the time integrals of the first internal node from the root of any component of D'' by zero (in the above example we replace the lower bound of the integral w.r.t. s from s_4 to 0) we get a factor that is bounded by $|m|^{-p}\tilde{E}_m\langle 1, \eta_t\rangle^k \mid_{D''}$, where $\tilde{E}_m\langle 1, \eta_t\rangle^k \mid_{D''}$ is the contribution to $\tilde{E}_m\langle 1, \eta_t\rangle^k$ coming from the graph D''. This leaves us with a graph that looks identical to D except for an additional p time integrals (created at the points of insertion of the p components of D'' into D).

Carrying out the estimation of the term as in the proof of Lemma 2.2 and using (2.20) we obtain

$$\tilde{E}_m\langle 1, \eta_t\rangle^k (J^\epsilon)^j \leq \tilde{E}_m\langle 1, \eta_t\rangle^k \tilde{E}_m(J^\epsilon)^j + \sum_{D\in\mathcal{D}_{2j}} \sum_{D''\in\mathcal{D}_k} \sum_{p=1}^{r} \binom{r}{p} \times$$

$$\times \sum_{l=1}^{p} \sum_{\Lambda(p,l)} \begin{pmatrix} p \\ p_1 \end{pmatrix} \begin{pmatrix} p - p_1 \\ p_2 \end{pmatrix} \cdots \begin{pmatrix} p - \ldots - p_{l-2} \\ p_{l-2} \end{pmatrix} p_1! \cdots p_l! \times$$

$$\times \sum_{i \in I} C(|m|, D, \epsilon, i) \frac{T^{h+p}}{(h+1)(h+2)\ldots(h+p)} \tilde{E}_m \langle 1, \eta_t \rangle^k \,|_{D''} \frac{1}{|m|^{r-p}}, \quad (2.23)$$

where $I = I(D, D'', p_1, \ldots, p_l)$ is the index set of all possible insertions of p out of the r components of D'' into the graph D, divided into groups with p_i of the p components in the i–th group, $i = 1, \ldots, l$. The $\begin{pmatrix} r \\ p \end{pmatrix}$ factor is the number of ways in which we can choose p components out of the r components. l is the number of groups into which we divide these p components. The subsequent $l-1$ combinatorial factors in (2.23) are the number of ways into which the p components can be divided into l groups. Components in each of these l groups is inserted into a distinct edge of D. After insertion the p_i components can be permuted in $p_i!$ ways, which accounts for the factors $p_1! \cdots p_l!$. Note again that we should have divided by the factorials of the number of identical p_i's, but since we are interested here in an upper bound, we ignore this factor. $h \equiv h(p, D) = -c(D)d/2 + a(D, d)$ is the same as in (2.3). The following lemma relates the constant $C(|m|, D, \epsilon, i)$ with the constant $C(m, D, d)$ of Lemma 2.2. Denote by I' the subset of I that contains all the insertions of the subgraphs of D'' which leave the number of twin pairs in D unchanged.

Lemma 2.3

$$C(|m|, |D|, \epsilon, i) \approx \begin{cases} C(m, D, d) & 1 \leq d \leq 3 \\ C(m, D, d) b(d, D, \epsilon) & 4 \leq d \leq 7 \text{ and } i \in I' \end{cases} \quad (2.24)$$

where $b(d, D, \epsilon)$ is as given in Lemma 2.2. When $i \in I - I'$,

$$C(|m|, |D|, \epsilon, i) \leq \begin{cases} C'(|m|, D, d) < \infty & d = 4, 5, \\ C(m, D, d) b(d, D, \epsilon) O(-\ln(\epsilon)) & d = 6, \\ C(m, D, d) b(d, D, \epsilon) O(\epsilon) & d = 7. \end{cases} \quad (2.25)$$

Proof of Lemma 2.3. If $1 \leq d \leq 3$, then in this case there is no divergence in ϵ. So we can replace all the epsilon by zero and carry out the calculations as described prior to the statement of Lemma 2.3. In case $4 \leq d \leq 7$, note that each twin pair contributes a factor $b(d, D, \epsilon)^{1/k(D)}$ to the divergence of the moment $\tilde{E}_m(J_T^\epsilon)^j$. When $i \in I'$, the number of twin pairs remains unaltered and the calculation as above gives the constant as in the statement of Lemma 2.3. When $i \in I - I'$, the number of twin pairs in the graph is reduced. When a twin pair is disturbed due to the insertion of some of the subgraphs of D'', the integrals associated with this twin pair will diverge

at a slower rate. To see this, consider the case when a component of D'' is inserted into the edge leading to the exit node labelled t_1, in a twin pair labelled t_1 and t_2. The twin pair which was contributing to the divergence of the moment at the rate $\epsilon^{-(d-4)/2}$ (see 2.9), now gives a term of the form

$$\frac{1}{(2\pi)^{-d/2}} \int_s^T \int_s^T dt_{2k-1} dt_{2k} 1_{\{|t_{2k}-t_{2k-1}|>\epsilon\}} \int_s^{t_1 \wedge t} ds_1 (t_{2k-1} + t_{2k} - 2s)^{-d/2} \tag{2.26}$$

which is $O(-\ln(\epsilon))$ for $d = 6$, $O(\epsilon^{-1/2})$ for $d = 7$ and stays bounded for $d \leq 5$. It is easily seen from the calculations that insetion of more components will lead to more integrals in (2.26). The result will be that the term will remain bounded in ϵ for $d \leq 7$. This proves Lemma 2.3.

Writing the result of the above Lemma succintly, we can say that $C(|m|, |D|, \epsilon, i) \leq C(m, D, d) b(d, D, \epsilon)$, for all $i \in I$. The cardinality of I is $|D|(|D|-1)\cdots(|D|-l+1)$. Using these two facts and simplifying, we can bound (2.23) by

$$\tilde{E}_m \langle 1, \eta_t \rangle^k (J^\epsilon)^j \leq \tilde{E}_m \langle 1, \eta_t \rangle^k \tilde{E}_m (J^\epsilon)^j + \sum_{D \in \mathcal{D}_{2j}} \sum_{D'' \in \mathcal{D}_k} \sum_{p=1}^{r} \binom{r}{p} p! \times \tag{2.27}$$

$$\times \sum_{l=1}^{p} \sum_{\Lambda(p,l)} C(m, D, d) b(d, D, \epsilon) T^{h+p} \frac{|D|(|D|-1)\cdots(|D|-l+1)}{(h+1)(h+2)\ldots(h+p)} \tilde{E}_m \langle 1, \eta_t \rangle^k \,|_{D''} \frac{1}{|m|^{r-p}},$$

where C, b and h are as in Lemma 2.2.

We now derive a bound for h. Our aim is to show that the fraction $|D|(|D|-1)\cdots(|D|-l+1)/h(h+1)\ldots(h+p)$, appearing in (2.27) is bounded by a constant depending only on k.

Claim: $j \leq h(d, D) \leq |D|$.

Proof of Claim: Since $d < 8$, $h \geq -4c(D) + a(D, d) = -4c(D) + |D| - 2k(D)$. Obviously $h(d, D) \leq |D|$. $k(D)$ equals zero when $1 \leq d \leq 3$, and equals the number of twin pairs for $4 \leq d \leq 7$. We divide the $2j$ exit nodes in D into three categories: The ones that form twin pairs, the ones that contribute to $c(D)$, and the rest. For each twin pair, we can associate three distinct edges in D. With each group of nodes that contribute one unit to $c(D)$, we can associate at least six edges (for it takes at least four exit nodes with no twin pairs to add to $c(D)$). With the remaining exit nodes we can associate at least one distinct edge. Thus we have,

$$\begin{aligned} |D| &\geq 6c(D) + 3k(D) + (2j - 4c(D) - 2k(D)) \\ &\geq 2j + 2c(D) + k(D) \end{aligned}$$

So,

$$\begin{aligned} h \geq -4c(D) + |D| - 2k(D) &\geq 2j - 2c(D) - k(D) \\ &\geq 2j - 2\frac{(j - k(D))}{2} - k(D) \geq j, \end{aligned}$$

where we used the fact that $c(D) \leq (j - k(D))/2$ (see proof of Lemma 2.2).

Using the above claim and the fact that $|D| \leq 4j$, we get the bound $|D|(|D|-1)\cdots(|D|-l+1)/h(h+1)\dots(h+p) \leq 4^k$, since $l \leq p \leq r \leq k$. This together with the inequalities $p! \leq k!$, $|\Lambda(p,l)| \leq \begin{pmatrix} p-1 \\ l-1 \end{pmatrix}$ in (2.27) yields the bound

$$\begin{aligned} \tilde{E}_m\langle 1,\eta_t\rangle^k (J^\epsilon)^j \ &\leq \ \tilde{E}_m\langle 1,\eta_t\rangle^k \tilde{E}_m (J^\epsilon)^j + (4C_1)^k\ k! \sum_{D\in\mathcal{D}_{2j}} \sum_{D''\in\mathcal{D}_k} C(m,D,d) b(d,D,\epsilon) T^h \times \\ &\times \sum_{p=1}^{r} \begin{pmatrix} r \\ p \end{pmatrix} \sum_{l=1}^{p} \begin{pmatrix} p-1 \\ l-1 \end{pmatrix} \tilde{E}_m\langle 1,\eta_t\rangle^k\ |_{D''}, \end{aligned} \tag{2.28}$$

where $C_1 = (1 \vee |m|^{-1})(1 \vee T)$. Since $\sum_{i=1}^k \begin{pmatrix} k \\ i \end{pmatrix} \leq 2^k$, we get

$$\begin{aligned} \tilde{E}_m\langle 1,\eta_t\rangle^k (J^\epsilon)^j \ &\leq \ \tilde{E}_m\langle 1,\eta_t\rangle^k \tilde{E}_m (J^\epsilon)^j + (16C_1)^k\ k!\ \tilde{E}_m\langle 1,\eta_t\rangle^k \tilde{E}_m (J^\epsilon)^j \\ &= \ (1 + C^k\ k!)\tilde{E}_m\langle 1,\eta_t\rangle^k \tilde{E}_m (J^\epsilon)^j. \end{aligned} \tag{2.29}$$

¿From (2.16), (2.17) and (2.29), we conclude that

$$\tilde{E}^{n,\epsilon,\lambda}_{m,T}\langle f,\eta_t\rangle^k \leq \|f\|^k (1 + k!C^k)\tilde{E}_m\langle 1,\eta_t\rangle^k. \tag{2.30}$$

Tightness of the marginals now follows from the boundedness of the moments.

To complete the proof, we show that for any positive function $f \in C_b^2(\Re^d)$, such that $\|f\| + \|\Delta f\| \leq 1$,

$$\tilde{E}^{n,\epsilon,\lambda}_{m,T}[\langle f,\eta_t\rangle - \langle f,\eta_s\rangle]^4 \leq C'_T(t-s)^2. \tag{2.31}$$

We know that (see Dawson (1993), Theorem 4.7.2, Proposition 7.3.1)

$$\tilde{E}_m[\langle f,\eta_t\rangle - \langle f,\eta_s\rangle]^4 \leq C_T(t-s)^2. \tag{2.32}$$

Note that we cannot apply the martingale approach of Proposition 7.3.1 from Dawson (1993), since the measure M appearing there is no longer a martingale under the new law. We can, however, follow the method employed in proving Theorem 4.7.2 there, which involves expanding the fourth power of the difference inside the expectation and then carrying out a direct computation, using the boundedness of f, the contraction property of the Brownian motion semigroup and the relation

$$\|T_t f - T_s f\| \approx \|\Delta f\|(t-s). \tag{2.33}$$

The calculation essentially involves a clever juggling of the ranges of integration of the time variable associated with the exit nodes. To analyse (2.31), expand the fourth power of the difference on the left after writing the expectation $\tilde{E}$ as a ratio. Each

term in the numerator involves evaluating a moment of order $2j+4$. We divide the graphs contributing to the $2j+4$ moments into two parts as done while evaluating $\tilde{E}_m\langle 1, \eta_t\rangle (J^\epsilon)^j$. The first part consists of attaching separate graphs from $\mathcal{D}_4$ to graphs from $\mathcal{D}_{2j}$. This will give a term $C_T(t-s)^2$. The second part consists of those graphs that can be constructed by inserting graphs from $\mathcal{D}_4$ into the edges of graphs from $\mathcal{D}_{2j}$ as done earlier in this proof. Consider the terms in the expansion that correspond to the same graph from $\mathcal{D}_{2j}$ into which a graph from $\mathcal{D}_4$ (this graph will be same for all the terms except that the time variable corresponding to the exit nodes will be different) is inserted. The factor coming from the graph from $\mathcal{D}_{2j}$ is common to all the terms and can be factored out. The inserted parts can now be analysed exactly as in the proof of Theorem 4.7.2 (Dawson, 1993). This gives us a factor $(t-s)^2$. The remaining terms can evaluated exactly as in the proof of Theorem 1.2. This will give us a bound for the required expectation of the form

$$(C_T + C(T,d,D,|m|))(t-s)^2 := C_T'(t-s)^2,$$

which completes the proof of tightness. ■

Proof of Theorem 1.3. a. So as to avoid some cumbersome integrals involving logarithms in even dimensions, we shall prove the result only for odd values of $d \le 7$. Filling in the even values requires patience, but no further ingenuity. We start with

$$J_T^\epsilon = 2\int_\epsilon^T dt_1 \int_0^{t_1} dt_2 \langle \delta(x-y), \eta_{t_1}(dx)\eta_{t_2}(dy)\rangle,$$

and first evaluate $\tilde{E}_m(J^\epsilon)$.

$$\tilde{E}_m(J^\epsilon) = 2\int_\epsilon^T dt_1 \int_0^{t_1} dt_2 \left[\int m(dx_1)m(dx_2)p_{t_1}(x_1,z)p_{t_2}(x_2,z)dz + \right.$$

$$\left. 2\int m(dx)p_s(x,y_1)p_{t_1-s}(y_1,z)p_{t_2-s}(y_1,z)\right] \tag{2.34}$$

The first term can be written as

$$2\int_\epsilon^T dt_1 \int_0^{t_1} dt_2 \langle m, T_{t_1+t_2} f\rangle = A_1^\epsilon(T), \tag{2.35}$$

where we denote by f the density of m w.r.t. the Lebesgue measure. The second term gives

$$4\int_\epsilon^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} ds |m|(t_1+t_2-2s)^{-d/2}$$

which equals

$$\frac{8|m|}{(d-2)(d-4)}\left[\frac{1}{(d-6)}\left\{4T^{-\frac{d-6}{2}} - 3\epsilon^{-\frac{d-6}{2}} - (2T-\epsilon)^{-\frac{d-6}{2}}\right\} + (T-\epsilon)\epsilon^{-\frac{d-4}{2}}\right]. \tag{2.36}$$

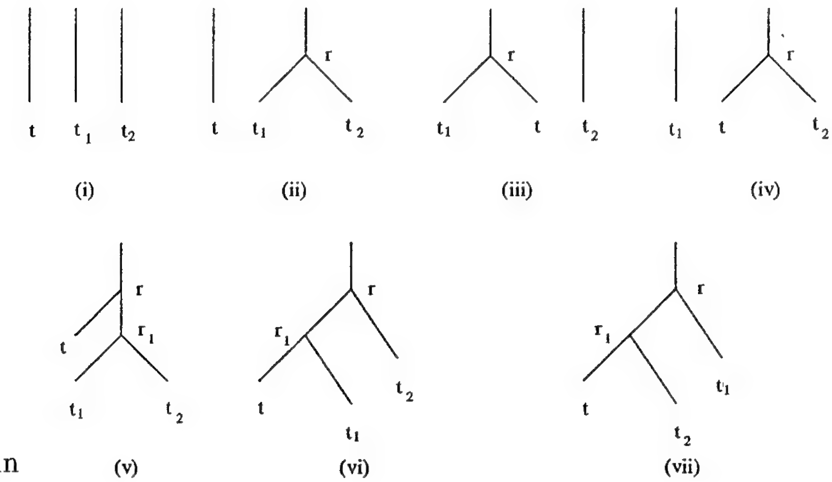

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Figure 4: A graph for the compuation of $E_m\langle 1,\eta_t\rangle J_T^\epsilon$.

We now proceed to evaluate $E_m\langle 1,\eta_t\rangle J_T^\epsilon$. Diagramatically, we can express this moment as in Figure 4. The first two graphs add to give $|m|E(J^\epsilon)$. The third graph corresponds to

$$4\int_\epsilon^T dt_1 \int_0^{t_1-\epsilon} dt_2 \int_0^{(t_1\wedge t)} ds \int m(dx)m(dy)p_s(x,y_1)p_{t-s}(y_1,w)p_{t_1-s}(y_1,z)p_{t_2}(y,z)dy_1dzdw$$

$$= 4\int_\epsilon^T dt_1 \int_0^{t_1-\epsilon} dt_2 \int_0^{(t_1\wedge t)} ds \int m(dx_1)m(dx_2)p_{t-1+t_2}(x_1,x_2) := A_2^\epsilon(t,T) \qquad (2.37)$$

Similarly, the fourth graph gives a contribution of $A_3^\epsilon(t,T)$. The fifth graph gives a contribution of

$$16|m|\left[\frac{1}{(d-2)(d-4)}\left\{\epsilon^{-\frac{d-4}{2}}\left[\frac{(T-\epsilon)^2}{2}-\frac{(T-t-\epsilon)^2}{2}\right]+\frac{1}{(d-6)}\left\{-3\epsilon^{-\frac{d-6}{2}}+\right.\right.\right.$$

$$\left.\left.\left.\frac{1}{(d-8)}\left\{[(2T-\epsilon)^{-\frac{d-8}{2}}-(2T-2t-\epsilon)^{-\frac{d-8}{2}}]+8[(T-t)^{-\frac{d-8}{2}}-T^{-\frac{d-8}{2}}]\right\}\right\}\right\}\right] \qquad (2.38)$$

The remaining terms can be similarly evaluated. The point to note here is that for $4 \le d \le 7$, the only terms that will matter in the $\epsilon \to 0$ limit are those of order $\epsilon^{-(d-4)/2}$. All the terms that we have not evaluated explicitly here will diverge at a slower rate. In the case $d \le 3$, all terms containing multiples of ϵ will vanish leaving a ratio that will look like the one given in the statement of the theorem. In this case terms coming from all the graphs will matter. Furthermore, since each of the integrals we evaluated is positive for all ϵ, the result follows.

b. To prove part (b), we will write a finer version of (2.23) for the case $k = 1$. See proof of part (a) above for illustrations. Fix $n \ge 1$. As $\epsilon \to 0$, only the terms with the

highest order of divergence in ϵ will be relevant. From Lemma 2.2 and (2.23), (2.27), we can write

$$\tilde{E}^{n,\epsilon,\lambda}_{m,T}\langle 1,\eta_t\rangle \approx \frac{\tilde{E}_m[\langle 1,\eta_t\rangle (J^\epsilon)^n]}{\tilde{E}_m(J^\epsilon)^n}, \tag{2.39}$$

as $\epsilon \longrightarrow 0$. As is now routine, we divide the numerator into a sum where a graph in $\mathcal{D}_{2n+1}$ is obtained by considering the graphs in $\mathcal{D}_{2n}$ and adding to them the graph in $\mathcal{D}_1$ as a seperate component. This on dividing by the denominator gives the term $\tilde{E}_m\langle 1,\eta_t\rangle = |m|$. We now focus on the second part. In course of the derivation of (2.23), if we use the fact that the effect of insertion also restricts the range of the time variables in the graph D, (see argument following (2.22)), then we would have ended with a factor $T^{h+p} - (T-t)^{h+p}$ instead of T^{h+p}. Further as $\epsilon \longrightarrow 0$, only those graphs in $\mathcal{D}_{2n}$, matter, for which there are n twin pairs (see Lemmas 2.2,2.3). We index the set of such graphs by $\mathcal{E}_{2n}$. For any $D \in \mathcal{E}_{2n}$, $k(D) = n$, and $C(D) = 0$. So $h(D) = |D| - 2n$. Now for the combinatorics part. The number of ways in which we can insert the only edge of the graph $D'' \in \mathcal{D}_1$ into a graph $D \in \mathcal{E}_{2n}$ so as not to disturb the number of twin pairs (see Lemma 2.3) is $|D| - 2n$. So, in the limit $\epsilon \longrightarrow 0$, (2.39) behaves like

$$|m| + \frac{|m|^{-1}\sum_{D\in\mathcal{E}_{2n}} C(m,D,d)b(d,D,\epsilon)(T^{h+1} - (T-t)^{h+1})\frac{|D|-2n}{|D|-2n+1}}{\sum_{D\in\mathcal{E}_{2n}} C(m,D,d)b(d,D,\epsilon)T^h}. \tag{2.40}$$

As $n \longrightarrow \infty$, $h = (|D| - 2n) \longrightarrow \infty$, since $3n \le |D| \le 4n$, and so (2.40) converges to $|m| + |m|^{-1}T$.

This proves Theorem 1.3. ■

Proof of Theorem 1.4. Using the same arguments as in the proof of Theorem 1.3, we can write

$$\tilde{E}^{n,\epsilon,\lambda}_{m,T}\langle 1,\eta_t\rangle^k \approx \frac{\tilde{E}_m[\langle 1,\eta_t\rangle^k (J^\epsilon)^n]}{\tilde{E}_m(J^\epsilon)^n}, \quad k \ge 1, \tag{2.41}$$

as $\epsilon \longrightarrow 0$. Again proceeding as in Theorem 1.3, we rewrite (2.23), by using the precise limits of integration while integrating the time variables. Then we simplify to get a form similar to (2.27. As above we note that as $\epsilon \longrightarrow 0$, only graphs in $\mathcal{E}_{2n}$ will matter. Now, in the $n \longrightarrow \infty$ limit, since $h \longrightarrow \infty$, the only combinatorial factor that will balance with the denomintor (see (2.27) are the ones for which $l = p$, that is $p_i = 1$, for $i = 1,\dots,p$, i.e.

$$\frac{|D|(|D|-1)\cdots(|D|-l+1)}{(|D|+1)(|D|+2)\cdots(|D|+p)} \longrightarrow \begin{cases} 1 & \text{if } l = p \\ 0 & \text{otherwise} \end{cases} \tag{2.42}$$

So, in the limiting procedure prescribed in the statement of the theorem, (2.41)

behaves like

$$\tilde{E}_m\langle 1,\eta_t\rangle^k + \frac{\sum_{D''\in\mathcal{D}_k}\tilde{E}_m\langle 1,\eta_t\rangle^k|_{D''}\sum_{p=1}^r P(T,t,p,D'')\frac{1}{|m|^{r-p}}\sum_{D\in\mathcal{E}_{2n}}C(m,D,d)b(d,D,\epsilon)T^h}{\sum_{D\in\mathcal{E}_{2n}}C(m,D,d)b(d,D,\epsilon)T^h}, \tag{2.43}$$

where $P(T,t,p,D'')$ is a polynomial of degree p in t,T. Thus we have proved that

$$\lim_{n\to\infty}\lim_{\epsilon\downarrow 0}\tilde{E}^{n,\epsilon,\lambda}_{m,T}\langle 1,\eta_t\rangle^k = \tilde{E}_m\langle 1,\eta_t\rangle^k + \sum_{D''\in\mathcal{D}_k}\tilde{E}_m\langle 1,\eta_t\rangle^k|_{D''}\sum_{p=1}^r P(T,t,p,D'')\frac{1}{|m|^{r-p}} \tag{2.44}$$

Repeating the above arguments, we can show that the mixed moments of the process X_t are unique under the given limits. This proves Theorem 1.4. ■

Proof of Theorem 1.5. It is clear that exponential integrability will follow once we get an appropriate bound on the the moments of $\langle f,\eta_t\rangle^k \le \|f\|_\infty\langle 1,\eta_t\rangle^k$. From (2.23), (2.27) and the arguments in Theorem 1.4, we have $P(T,t,p,D'') \le P(T,T,p,D'') = T^p \le (1\vee T)^k$. $P(T,t,p,D'')\frac{1}{|m|^{r-p}} \le C_1^k$, where $C_1 = (1\vee T)(1\vee|m|^{-1})$. This together with (2.44) gives

$$\lim_{n\to\infty}\lim_{\epsilon\downarrow 0}\tilde{E}^{n,\epsilon,\lambda}_{m,T}\langle 1,\eta_t\rangle^k \le \tilde{E}_m\langle 1,\eta_t\rangle^k(1+(2C_1)^k) \le C^k\tilde{E}_m\langle 1,\eta_t\rangle^k. \tag{2.45}$$

This implies that

$$\lim_{n\to\infty}\lim_{\epsilon\downarrow 0}\tilde{E}^{n,\epsilon,\lambda}_{m,T}\langle f,\eta_t\rangle^k \le (C\|f\|)^k\tilde{E}_m\langle 1,\eta_t\rangle^k \tag{2.46}$$

The result now follows from the exponential integrability of the Feller diffusion; i.e. from the fact that $\tilde{E}_m e^{\theta\langle 1,\eta_t\rangle}$ exists and is finite for all $t < \theta^{-1}$.

Proof of Theorem 1.6. By definition,

$$E^\lambda_{x,T,\alpha}(e^{-cX_t}) = \frac{E_{x,T,\alpha}(e^{-cX_t+\lambda\mu_T})}{E_{x,T,\alpha}(e^{\lambda\mu_T})}. \tag{2.47}$$

Let $v_t^{\alpha,\lambda,c}$ denote the unique solution of the following differential equation:

$$\frac{dv_t}{dt} = \alpha v_t - v^2 - \lambda\ ;\ v_0 = c. \tag{2.48}$$

Iscoe (1986) has shown that

$$E_{m,T,\alpha}(e^{-\langle f,\eta_t\rangle+\lambda\int_0^t\langle\phi,\eta_s\rangle ds}) = e^{\langle u_t^{\alpha,\lambda,f},m\rangle}, \tag{2.49}$$

where $u_t^{\alpha,\lambda,f}$ is the unique solution of the non-linear initial value problem:

$$\begin{aligned} \frac{\delta u_t}{\delta t} &= (\Delta + \alpha)u_t - u^2 - \lambda\phi \\ u_0 &= f. \end{aligned} \tag{2.50}$$

When the functions f and ϕ are constants, then u no longer has a spatial dependence. The denominator in (2.47) can be written as $\exp(-xv_T^{\alpha,\lambda,0})$. Using the Markov property and homogeneity the numerator can be seen to be

$$\exp(-xv_t^{\alpha,\lambda,c+v_{T-t}^{\alpha,\lambda,0}}). \tag{2.51}$$

Proving Theorem 1.6 now involves verifying that the denominator and the numerator in the various cases satisfy the requisite differential equation. For example, the denominator in case (i) on the right can be easily seen to be $\exp(-xv_T^{0,\lambda,0})$. We leave the straightforward verification of the other cases to the enthusiastic reader and so conclude the proof of Theorem 1.6. ■

Proof of Corollary 1.7. The corollary follows immediately by noting that in each case, as T approaches the critical value, the denomiator tends to $+\infty$, while the numerator stays bounded. ■

3 Some results for the reinforced Feller diffusion

Any number of moment formulae and relationships can be gleaned from the formulae in Theorem 1.6. Here are a few illustrative examples.

Corollary 3.1 *Let γ be as in Theorem 1.6. Let $\lambda > 0$.*

(i)

$$E_{x,T,0}^{\lambda}(X_t) = \frac{x\ sec^2(\sqrt{\lambda}T)}{sec^2(\sqrt{\lambda}(T-t))}. \tag{3.1}$$

(ii)

$$Var_{x,T,0}^{\lambda}(X_t) = \frac{2x}{\sqrt{\lambda}} \frac{sec^2(\sqrt{\lambda}\,t)[\tan(\sqrt{\lambda}\,T) - \tan(\sqrt{\lambda}(T-t))]}{sec^4(\sqrt{\lambda}(T-t))} \tag{3.2}$$

(iii) If $\gamma = 0$, then

$$E_{x,T,\alpha}^{\lambda}(X_t) = \frac{(1-\frac{\alpha}{2}(T-t))x}{(1-\frac{\alpha}{2}T)} \tag{3.3}$$

(iv) If $\gamma = 0$, and $\alpha < 0$, then

$$\lim_{T\to\infty} E^{\lambda}_{x,T,\alpha}(e^{-cX_t}) \;=\; E_x(e^{-cX_t}) = e^{-cx/(1+ct)}, \tag{3.4}$$

where the right side of the equality is the Laplace transform of a Feller diffusion.

(v) Let $\alpha = -2\sqrt{\lambda}$. Then,

$$\lim_{T\to\infty} E^{-\lambda}_{x,T,0}(X_t) \;=\; E_{x,\infty,\alpha}(X_t) \;=\; xe^{\alpha t}. \tag{3.5}$$

Remark: Result (v) says that, depending on λ and α, the expected reduction in mass in time for the reinforced Feller diffusion and a simple Feller diffusion with a mass annihilation term are almost identical for large T. The distributions in the two cases are, however, quite different.

Proof of Corollary 3.1. The corollary is an immediate consequence of Theorem 1.6 and the equalities

$$E^{\lambda}_{x,T,\alpha}X_t = -\frac{d}{dc}E^{\lambda}_{x,T,\alpha}(e^{-cX_t})\mid_{c=0} \tag{3.6}$$

and

$$E^{\lambda}_{x,T,\alpha}X_t^2 = \frac{d^2}{dc^2}E^{\lambda}_{x,T,\alpha}(e^{-cX_t})\mid_{c=0} \tag{3.7}$$

■

Corollary 3.2 *(Extinction Probabilities) Let $\lambda > 0$ and let γ be as above.*

(i)

$$P_{x,T,\alpha}(X_t = 0) \;=\; \exp\left\{\frac{\alpha x e^{\alpha t}}{1 - e^{\alpha t}}\right\}. \tag{3.8}$$

(ii) If $\gamma > 0$ and $0 < T < \frac{1}{\sqrt{\gamma}}\left(\frac{\pi}{2} - \tan^{-1}(\frac{\alpha}{2\sqrt{\gamma}})\right)$, then

$$P^{\lambda}_{x,T,\alpha}(X_t = 0) \;=\; \exp\{-\sqrt{\gamma}x[\tan(\frac{\pi}{2} - \sqrt{\gamma}t) + \tan(\tan^{-1}(\frac{\alpha}{2\sqrt{\gamma}}) + \sqrt{\gamma}T)]\}. \tag{3.9}$$

(iii) If $\gamma = 0$ and $0 < T < 2/\alpha$ if $\alpha > 0$, then

$$P^{\lambda}_{x,T,\alpha}(X_t = 0) \;=\; e^{-x\left[\frac{2-\alpha(T-t)}{t(2-\alpha T)}\right]} \tag{3.10}$$

(iv) If $\gamma < 0$ let $\beta = \sqrt{-\gamma}$. Then

$$P^{\lambda}_{x,T,\alpha}(X_t = 0) \;=\; \exp\left\{-\beta x\left[\frac{1 - e^{2\beta t}}{1 - e^{2\beta t}} + \frac{(\beta - \frac{\alpha}{2})e^{2\beta T} - (\beta + \frac{\alpha}{2})}{(\beta - \frac{\alpha}{2})e^{2\beta T} + (\beta + \frac{\alpha}{2})}\right]\right\} \tag{3.11}$$

Proof of Corollary 3.2 The corollary follows from Theorem 1.6 and the equality

$$P^{\lambda}_{x,T,\alpha}(X_t = 0) = \lim_{c \to \infty} E^{\lambda}_{x,T,\alpha}(e^{-cX_t}) \tag{3.12}$$

■

4 Appendix

For completeness, we shall now discuss the finiteness of the moments of the approximate self intersection local times J^ϵ. While this result is similar to others that have appeared in the literature, it is, in fact new.

Theorem 4.1 *$\tilde{E}(J^\epsilon)^j < \infty$ for all $j \geq 1$, $T < \infty$ and $d \leq 7$.*

Proof. We only give a sketch of the proof. The reader is referred to Dynkin (1988), Rosen (1992) and Feldman and Iyer (1996), where similar results can be found, for the details on evaluating moments similar to the one below.

Let $R^B_\epsilon(s_1, s_2, y_1 - y_2) = \int_B dt_1 dt_2 p_{t_1+t_2-s_1-s_2}(y_1 - y_2)$, where $B = [0,T]^2 \cap \{(t_1, t_2) : |t_1 - t_2| > \epsilon\}$. Recall that

$$J^\epsilon = \int_B \langle \delta(x-y), \eta_{t_1}(dx)\eta_{t_2}(dy)\rangle dt_1 dt_2,$$

so we can write

$$(J^\epsilon)^j = \prod_{i=1}^{j} \int_B \langle \delta(z_{2i-1} - z_{2i}), \eta_{t_{2i-1}}(dz_{2i-1})\eta_{t_{2i}}(dz_{2i})\rangle dt_{2i-1} dt_{2i}.$$

$E(J^\epsilon)^j = \sum_{D \in \mathcal{D}_{2j}} C_D$. Consider a term C_D. We separate our analysis of C_D into the following cases:

(1) If the exit nodes labelled (t_{2i-1}, z_{2i-1}) and (t_{2i}, z_{2i}) are paired, that is they have an immediate common ancestor labelled (s_k, y_k), then integrating w.r.t. z_{2i} and z_{2i-1} and using the fact that

$$\int_{\Re^{2d}} \delta(z_1 - z_2) f(z_1) g(z_2) dz_1 dz_2 = \int_{\Re^d} f(w) g(w) dw, \tag{4.1}$$

we obtain a factor

$$R^B_\epsilon(s_k, s_k, 0) = \int_B dt_{2i-1} dt_{2i} p_{t_{2i-1}+t_{2i}-2s_k} < c|B|\epsilon^{-d/2}. \tag{4.2}$$

So we can bound the contribution coming from all the paired nodes.

(2) If at least one of the exit nodes z_{2i-1} or z_{2i} is connected to a root (entrance) node labelled $(0, x_l)$, then we get a contribution $\int m(dx) \int_B dt_{2i-1} dt_{2i} p_{t_{2i-1}+t_{2i}-s_k}(y_k - x_l)$. Using the bounded density of m we integrate out x_l to see that the above term is finite.

(3) Now, of the remaining exit nodes whose space variables z_i's are left to be integrated, none of them are paired or are connected to a root node. (These have been taken care of in the above two cases). Suppose there are $2r$ remaining z_i's. If z_{2i} and z_{2i-1} are connected to the internal nodes labelled (s_k, y_k) and (s_l, y_l), then integrating with respect to z_{2i} and z_{2i-1} using (4.1) gives a factor $R(s_k, s_l, y_k - y_l)$. Since there are $2r$ z_i's to be integrated, we will get r factors that look like $R(s_k, s_l, y_k - y_l)$. Each (s_k, y_k) occurs in exactly 2 of the $R's$. Make suitable change of variables (see Feldman and Iyer, 1996, for details) $w_i = y_k - y_l$, such that $\{y_{i_1}, \ldots, y_{i_p}, w_{i_1}, \ldots, w_{2r-p}\}$ spans $\Re^{2rd}$. Since each function R in the integrand has an argument that is a difference of the y's, upon change of variables all the R's will have arguments depending only on the w's. Furthermore, each w occurs in exactly two of the R's. Also to be noted is the fact that one cannot have $w_i - w_j$ occuring as the sole argument in two of the R's.

First integrate out all the variables $y_{i_1}, \ldots, y_{i_p}$. Now using the bounded density of m, if necessary, one can integrate out all the internal space variables. This will leave us with the factors R and the time integrals only. Now we integrate with respect to the w's. If one has a factor $\int R^2(s_1, s_2, w_1) dw_1 ds_1 ds_2$, it can be written as

$$\begin{aligned} \int ds_1 ds_2 \int_{B^2} p_{t_1+t_2+t_3+t_4-2s_1-2s_2}(0) \prod_{i=1}^{4} dt_i &< T^2 \int_{[0,T]^4} (t_1 + t_2 + t_3 + t_4)^{-d/2} \prod_{i=1}^{4} dt_i \\ &< \infty \text{ for } d \leq 7. \end{aligned} \tag{4.3}$$

A factor of the form

$$\int ds_1 ds_2 ds_3 \int R(s_1, s_2, w_1) R(s_1, s_3, w_2) R(s_2, s_3, w_2 - w_1) dw_1 dw_2 dw_3 \tag{4.4}$$

arising out of the configuration shown in Figure 5 is bounded by $T^3 \int_{[0,T]^6} (t_1 + \ldots + t_6)^{-d/2} dt_1 \ldots dt_6$ is finite for $d \leq 11$. Similarly we can check that the more one has to integrate w.r.t. the w's, the smoother the integral becomes. the worst case being given by (4.3). We conclude the proof of Theorem 4.1. ■

References

[1] Adler, R.J. (1992) Superprocess local and intersection local times and their

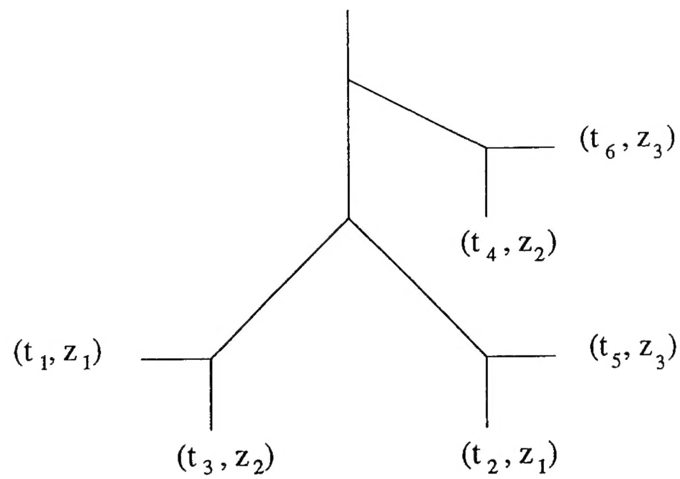

Figure 5: Computing the factor (4.4).

corresponding particle pictures. *Seminar on Stochastic Processes, 1992.* eds E. Çinlar, K.L. Chung, M.J. Sharpe, Birkhauser, 1–42.

[2] Adler, R.J. and Ivanitskaya, L. (1996) A superprocess with a disappearing self-interaction, *J. Theoretical Probability*, **9** 245-261.

[3] Adler, R.J. and Iyer, S.K. (1997) Attractive polymer models for two and three dimensional Brownian motion. *Stoc. Proc. Appl.* **66** 271-281.

[4] Adler, R.J. and Tribe, R. (1996a) Uniqueness for a historical SDE with a singular interaction. Submitted.

[5] Adler, R.J. and Tribe, R. (1996b) On a one-dimensional historical SDE with a singular interaction. In preparation.

[6] Dawson, D. (1978), Geostochastic calculus, *Canadian J. Statistics*, **6**, 143–168.

[7] Dawson, D. (1993) Measure valued Markov processes, in *Ecole d'Eté de Probabilités de Saint-Flour, XXI. Lect. Notes Math.* **1541** Springer, 1–260.

[8] Dawson, D. and Perkins, E.A. (1991), Historical processes, *Amer Math Soc Memoirs*, **93** 1–179.

[9] Dynkin, E.B. (1988) Representation for functionals of superprocesses by multiple stochastic integrals, with applications to self intersection local times, *Astérisque*, **157–158**, 147–171.

[10] Feldman, R.E. and Iyer, S.K. (1996) A representation for functionals of superprocesses via particle pictures, *Stoch. Proc. Appl.* **64** 173–186.

[11] Iscoe, I. (1986a), A weighted occupation time for a class of measure valued branching processes, *Z. Wahrsch. verw. Gebeite* **71** 85–116.

[12] Iscoe, I. (1986b), Ergodic theory and a local occupation time for measure valued critical branching Brownian motion, *Stochastics* **18** 197–243.

[13] Perkins, E. (1992) Measure valued branching diffusions with spatial interactions, *Probab. Theory and Related Fields* **94** 189–245.

[14] Rosen, J. (1992) Renormalization and limit theorems for self-intersections of superprocesses, *Ann. Probability* **20** 1341–1368.

[15] Varadhan, S.R.S. (1969), Appendix to Euclidean quantum field theory by K. Symanzik. In: Jost, R. (ed.)*Local Quantum theory*, Academic, New York.

[16] Westwater, J. (1980), On Edwards' model for polymer chains, *Comm. Math. Phys.* **72** 131–174.

[17] Westwater, J. (1982), On Edwards' model for polymer chains III, Borel summability, *Comm. Math. Phys.* **84** 459–470.

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